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A probabilistic approach to special values of the Riemann zeta function [1]

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Abstract: In this paper, using Cauchy variables, we get a new elementary proof of $\zeta(2)=\frac{\pi^2}{6}$. Furthermore, as its generalization, using variants of Cauchy variables, we get further results about ζ. We also get two different proofs of Euler's formulae for the Riemann zeta function via independent products of Cauchy variables. This paper is a review of our previous papers ([3, 4]).
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1 Introduction

Consider the Riemann zeta function

$$\zeta(s)=\sum_{j=1}^{\infty}\frac{1}{j^s} \qquad (\text{for} \quad \mathrm{Re}\, s>1).$$

Many authors ([1, 6, 8, 9, 10, 12]) have written elementary proofs of $\zeta(2)=\frac{\pi^2}{6}$. The problem of finding this value is known as Basel problem ([5]). First, in this paper, we propose a new elementary probabilistic proof of this famous result, using Cauchy variables.
After this, we investigate the following Euler's formulae of the Riemann zeta function, which is very classical (see for example [11]):

Euler's Formulae $$(1-\frac{1}{2^{2n+2}})\zeta(2n+2)=\frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+2}\frac{A_n}{\Gamma(2n+2)}.$$

Here, the coefficients A_n are featured in the series development

$$\frac{1}{\cos^2\theta}=\sum_{n=0}^{\infty}\frac{A_n}{(2n)!}\theta^{2n} \quad (|\theta|<\frac{\pi}{2}).$$

The most popular ways to prove Euler formula make use of Fourier inversion and Parseval's theorem, or of nontrivial expansions of such as cotan. In this

[1] This paper is a review version of our previous paper [3, 4].

paper, we show that Euler formula is obtained by simply via either of the following methods: In section 3, we compute in two different ways the moments $E((\log \mathbb{C}_1\mathbb{C}_2)^{2n})$ with $\mathbb{C}_1$ and $\mathbb{C}_2$ two independent standard Cauchy variables.

• One one hand, these moments can be computed explicitly in terms of ζ thanks to explicit formulae for the density $\mathbb{C}_1\mathbb{C}_2$.

• On the other hand, these moments are obtained via the representation $E(|\mathbb{C}_1|^{\frac{2i\lambda}{\pi}}) = \frac{1}{\cosh \lambda} \qquad (\lambda \in \mathbb{R})$.

In section 4, we derive the formulae for $\zeta(2n)$ from the explicit calculation of the density of the product $\mathbb{C}_1\mathbb{C}_2\cdots\mathbb{C}_k$ of k independent standard Cauchy variables, by expoiting the fact that the integral of this density is equal to 1. In section 5, we get further results involving $\sum_{k=0}^{\infty}\frac{1}{(mk+n)^2}$ using variants of Cauchy variables, also derived in an elementary manner and made several remarks.

2 Basel Problem and Products of Independent Cauchy variables

a) We first review the density function of the multiplicative convolution of two independent random variables from elementary probability theory.

Lemma2.1.
Consider two independent random variables X, Y such that $P(X>0) = P(Y>0) = 1$ and with density functions $f_X(x)$, $f_Y(x)$.

Then, $f_{XY}(x)$, the density function of XY is given by:

$$f_{XY}(x) = \int_0^{\infty} f_X(u) f_Y(\frac{x}{u})\frac{1}{u}du$$

Proof.

For $x>0$, $P(XY<x) = \int\int_{uv<x} f_X(u)f_Y(v)dudv = \int_0^{\infty} f_X(u)du \int_0^{\frac{x}{u}} f_Y(v)dv$.
Then differentiating both sides with respect to x, we get the result.
b) Applying Lemma 2.1. for $f_X(x) = f_Y(y) = \frac{2}{\pi}\frac{1}{1+x^2}1_{x>0}$ i.e.: $X \sim Y \sim |C|$ where C is a Cauchy variable with $f_C(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, we get , for $x>0$:

$$\begin{aligned}
f_{XY}(x) &= \frac{4}{\pi^2}\int_0^{\infty}\frac{1}{(1+u^2)}\frac{1}{(1+(\frac{x}{u})^2)}\frac{1}{u}du \\
&= \frac{2}{\pi^2}\int_0^{\infty}\frac{1}{(u+1)(u+x^2)}du \\
&= \frac{2}{\pi^2}\int_0^{\infty}(\frac{1}{u+x^2}-\frac{1}{u+1})\frac{du}{1-x^2} \\
&= \lim_{A\to\infty}\frac{2}{\pi^2}\int_0^{A}(\frac{1}{u+x^2}-\frac{1}{u+1})\frac{du}{1-x^2} \\
&= \frac{4}{\pi^2}\frac{\log x}{x^2-1}
\end{aligned}$$

c) Since $1 = \int_0^\infty f_{XY}(x)dx$, we have:

$$\frac{\pi^2}{4} = \int_0^\infty \frac{\log x}{x^2-1}dx$$

The righthand side R is equal to

$$\begin{aligned}
R &= \int_0^1 \frac{\log x}{x^2-1}dx + \int_1^\infty \frac{\log x}{x^2-1}dx \\
&= 2\int_0^1 \frac{-\log x}{1-x^2}dx = 2\int_0^1 (-\log x)\sum_{k=0}^{\infty} x^{2k}dx \\
&= 2\sum_{k=0}^{\infty}\int_0^1 (-\log x)x^{2k}dx = 2\sum_{k=0}^{\infty}\int_0^\infty ue^{-2ku}e^{-u}du \\
&= 2\sum_{k=0}^{\infty}\int_0^\infty \frac{y}{2k+1}e^{-y}\frac{dy}{2k+1} = 2\Gamma(2)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}
\end{aligned}$$

Thus, we have obtained:

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Noting that $\zeta(2) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} + \frac{1}{2^2}\zeta(2)$,
we obtain the desired result, i.e.:

$$\zeta(2) = \frac{4}{3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}\frac{4}{3} = \frac{\pi^2}{6}.$$

This is a probabilistic solution of Basel Problem.

3 Euler's Formulae via Cauchy variables

In this section, considring even moments of $\log \mathbb{C}_1\mathbb{C}_2$, we prove the Euler's formulae of the Riemann zeta function.

Proposition 3.1.

$$E((\log \mathbb{C}_1\mathbb{C}_2)^{2n}) = \frac{8}{\pi^2}\Gamma(2n+2)(1-\frac{1}{2^{2n+2}})\zeta(2n+2).$$

Proof.

The proof relies on the same computation as section 1.

Using this proposition, we obtain the following Euler Formula:
Theorem 3.2. **(Euler's Formulae)**

$$(1-\frac{1}{2^{2n+2}})\zeta(2n+2)=\frac{1}{2}\left(\frac{\pi}{2}\right)^{2n+2}\frac{A_n}{\Gamma(2n+2)}$$

where, the coefficients A_n are obtained in the series development

$$\frac{1}{\cos^2\theta}=\sum_{n=0}^{\infty}\frac{A_n}{(2n)!}\theta^{2n}\quad(|\theta|<\frac{\pi}{2}).$$

Proof.

We only need to prove that $E(|\mathbb{C}_1|^{\frac{2i\lambda}{\pi}})=\frac{1}{\cosh\lambda}$, because by this, we can easily get that $E(e^{i\lambda\frac{2}{\pi}\log|\mathbb{C}_1\mathbb{C}_2|})=\frac{1}{(\cosh\lambda)^2}$, which is equivalent to $E((\log|\mathbb{C}_1\mathbb{C}_2|)^{2n})=(\frac{\pi}{2})^{2n}A_n$.
Noting that $\mathbb{C}_1\sim\frac{N}{N'}$ where N and N' are two independent standard normal random variables, we get that $(\mathbb{C}_1)^2\sim\frac{N^2}{(N')^2}\sim\frac{\gamma_{1/2}}{\gamma'_{1/2}}$ where $\gamma_{1/2}$ and $\gamma'_{1/2}$ are two independent gamma variables with parameter 1/2, i.e. its density $f_{\gamma_{1/2}}(x)=\frac{x^{-1/2}}{\sqrt{\pi}}e^{-x}\quad(x>0)$.
Then we get that
$E(|\mathbb{C}_1|^{\frac{2i\lambda}{\pi}})=E((\gamma_{1/2})^{\frac{i\lambda}{\pi}})E((\gamma_{1/2})^{\frac{-i\lambda}{\pi}})=\frac{\Gamma(\frac{1}{2}+\frac{i\lambda}{\pi})}{\Gamma(1/2)}\frac{\Gamma(\frac{1}{2}-\frac{i\lambda}{\pi})}{\Gamma(1/2)}=\frac{1}{\pi}\frac{\pi}{\sin\pi(\frac{1}{2}+\frac{i\lambda}{\pi})}=\frac{1}{\cosh\lambda}\quad(\lambda\in\mathbb{R})$, where we used the fact : $\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}$.

$\square$

Remark3.3.
In [3], furthermore, we prove the L_{χ_4} case considering $E((\log\mathbb{C}_1)^{2n})$.

4 Explcit density function of independent products of Cauchy variables and the Riemann zeta function

In this section, we give the density of the law of $\mathbb{C}_1\mathbb{C}_2\cdots\mathbb{C}_k$ for any $k\geqq 0$, where $\mathbb{C}_1,\ldots\mathbb{C}_k$ are k independent Cauchy variables.

Proposition 4.1.

The density of $\mathbb{C}_1\mathbb{C}_2\cdots\mathbb{C}_k$ *is equal to*

$$f_{\mathbb{C}_1\mathbb{C}_2\cdots\mathbb{C}_{2n}}(x)=\frac{2^{2n-1}}{\pi^2(2n-1)!}\left(\prod_{j=1}^{n-1}(j^2+\frac{(\log|x|)^2}{\pi^2})\right)\frac{\log|x|}{x^2-1}$$

$$f_{\mathbb{C}_1\mathbb{C}_2\cdots\mathbb{C}_{2n+1}}(x)=\frac{2^{2n}}{\pi(2n)!}\left(\prod_{j=1}^{n}(j^2+\frac{(\log|x|)^2}{\pi^2})\right)\frac{1}{1+x^2}$$

Proof.

First we note that $E(|\mathbb{C}_1|^\alpha) = \frac{1}{\cos\frac{\pi\alpha}{2}}$ and $E(|\mathbb{C}_1\cdots\mathbb{C}_k|^\alpha) = \frac{1}{(\cos\frac{\pi\alpha}{2})^k}$ $(|\alpha| < 1)$.

Then we get that $E(|\mathbb{C}_1\cdots\mathbb{C}_k|^\alpha \log|\mathbb{C}_1\cdots\mathbb{C}_k|) = (-k)(\cos\frac{\pi\alpha}{2})^{-k-1}(-\sin\frac{\pi\alpha}{2})\frac{\pi}{2}$ and $E(|\mathbb{C}_1\cdots\mathbb{C}_k|^\alpha(\log|\mathbb{C}_1\cdots\mathbb{C}_k|)^2) = (\frac{\pi}{2})^2(k(k+1)(\cos\frac{\pi\alpha}{2})^{-k-2} - k^2(\cos\frac{\pi\alpha}{2})^{-k})$ $= (\frac{\pi}{2})^2(k(k+1)E(|\mathbb{C}_1\cdots\mathbb{C}_{k+2}|^\alpha) - k^2E(|\mathbb{C}_1\cdots\mathbb{C}_k|^\alpha)$.

Then by the uniqueness of Melin transformation, we get that $f_{\mathbb{C}_1\mathbb{C}_2\cdots\mathbb{C}_{k+2}}(x) = \frac{4}{k(k+1)}(\frac{(\log|x|)^2}{\pi^2} + \frac{k^2}{4})f_{\mathbb{C}_1\mathbb{C}_2\cdots\mathbb{C}_k}(x)$. This gives the results.

□

We note that $1 = \int_{-\infty}^{\infty} f_{\mathbb{C}_1\mathbb{C}_2\cdots\mathbb{C}_{2n}}(x)dx$ gives the recurrence relation between $\zeta(2n)$'s and we see that this recurrence relation is equivalent to Euler formulae. (see [3].) Furthermore, in [3], using $1 = \int_{-\infty}^{\infty} f_{\mathbb{C}_1\mathbb{C}_2\cdots\mathbb{C}_{2n+1}}(x)dx$, we prove the the L_{χ_4} case.

5 A two-parameter generalization and remarks

a) In order to generalize the former arguments, we take $f_{X_{m,n}}(x) = c_{m,n}\frac{x^m}{1+x^n}1_{x>0}$ for $n > m+1$ instead of the Cauchy density. In Remark 5.2., We realize $X_{m,n}$ as a power of the ratio of two independent gamma variables.
n and m are not assumed to be integers, although, for the applications, the integer case is often most interesting.

Putting $u = \frac{1}{1+x^n}$, we get :

$$\begin{aligned}
\int_0^\infty \frac{x^m}{x^n+1}dx &= \frac{1}{n}\int_0^1 u^{-\frac{m+1}{n}}(1-u)^{\frac{m+1}{n}-1}du \\
&= \frac{1}{n}B(1-\frac{m+1}{n}, \frac{m+1}{n}) = \frac{1}{n}\Gamma(1-\frac{m+1}{n})\Gamma(\frac{m+1}{n}) \\
&= \frac{\frac{\pi}{n}}{\sin\frac{m+1}{n}\pi}
\end{aligned}$$

where we used the formula of complements $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}$.

Thus, the normalizing constant $c_{m,n}$ is given by : $c_{m,n} = \frac{\sin\frac{m+1}{n}\pi}{\frac{\pi}{n}}$.

Similarly, if Y is an independent copy of $X = X_{m,n}$, then for $x > 0$:

$$\begin{aligned}
f_{XY}(x) &= c_{m,n}^2\int_0^\infty \frac{u^m}{1+u^n}\frac{(\frac{x}{u})^m}{1+(\frac{x}{u})^n}\frac{1}{u}du \\
&= \frac{c_{m,n}^2}{n}\int_0^\infty \frac{x^m}{(u+1)(u+x^n)}du \\
&= c_{m,n}^2\frac{x^m\log x}{x^n-1}
\end{aligned}$$

b) Again, using: $1 = \int_0^\infty f_{XY}(x)dx$, we obtain

$$
\begin{aligned}
\left(\frac{\frac{\pi}{n}}{\sin\frac{m+1}{n}\pi}\right)^2 &= \int_0^\infty \frac{x^m \log x}{x^n - 1}dx \\
&= \int_0^1 \frac{x^m \log x}{x^2 - 1}dx + \int_1^\infty \frac{x^m \log x}{x^n - 1}dx \\
&= \int_0^1 \frac{-x^m \log x}{1 - x^n}dx + \int_0^1 \frac{-u^{n-m-2}\log u}{1 - u^n}du \\
&= \sum_{k=0}^{\infty} \frac{1}{(nk+m+1)^2} + \sum_{k=0}^{\infty} \frac{1}{(nk+n-m-1)^2}
\end{aligned}
$$

Then we get the following :

Theorem 5.1.

For $n > m+1$:

$$
\sum_{k=0}^{\infty} \frac{1}{(nk+m+1)^2} + \sum_{k=0}^{\infty} \frac{1}{(nk+n-m-1)^2} = \left(\frac{\frac{\pi}{n}}{\sin\frac{m+1}{n}\pi}\right)^2. \quad (1)
$$

which is equivalent to:

$$
\sum_{k=-\infty}^{\infty} \frac{1}{(nk+m+1)^2} = \left(\frac{\frac{\pi}{n}}{\sin\frac{m+1}{n}\pi}\right)^2. \quad (2)
$$

2

c) The following examples of (1) are interesting:

$$
\sum_{k=0}^{\infty} \frac{1}{(5k+1)^2} + \sum_{k=0}^{\infty} \frac{1}{(5k+4)^2} = \left(\frac{\frac{\pi}{5}}{\sin\frac{1}{5}\pi}\right)^2.
$$

$$
\sum_{k=0}^{\infty} \frac{1}{(5k+2)^2} + \sum_{k=0}^{\infty} \frac{1}{(5k+3)^2} = \left(\frac{\frac{\pi}{5}}{\sin\frac{2}{5}\pi}\right)^2.
$$

Combining these, we get :

$$
\frac{1}{(\sin\frac{1}{5}\pi)^2} + \frac{1}{(\sin\frac{2}{5}\pi)^2} = 4
$$

Similarly

$$
\frac{1}{(\sin\frac{1}{7}\pi)^2} + \frac{1}{(\sin\frac{2}{7}\pi)^2} + \frac{1}{(\sin\frac{3}{7}\pi)^2} = 8
$$

[2] In the case of $m = 0$, K. Yano and Y. Yano obtained the higher moments of this equality similar to Euler's formulae.

More generally we obtain in this way:

$$\sum_{k=1}^{n} \frac{1}{(\sin \frac{k}{2n+1}\pi)^2} = \frac{2n(n+1)}{3}.$$

We note that in [1, 6], a proof of this formula by trigonometric arguments led to $\zeta(2) = \frac{\pi^2}{6}$. In [2], this formula is also obtained by considering the Parseval identity of some finite Fourier series.

Remark 5.2. Using two independent gamma variables $\gamma_{\frac{m+1}{n}}, \gamma'_{1-\frac{m+1}{n}}$, it is easily found that:

$$X_{m,n} \overset{(law)}{=} \left(\frac{\gamma_{\frac{m+1}{n}}}{\gamma'_{1-\frac{m+1}{n}}} \right)^{1/n}$$

where $f_{\gamma_a}(x) = \frac{x^{a-1}}{\Gamma(a)} e^{-x} 1_{x>0}$.

Remark 5.3. We see easily that formula (2) is equivalent to

$$\sum_{j=-\infty}^{\infty} \frac{1}{(j+x)^2} = \frac{\pi^2}{\sin^2(x\pi)} \qquad (3)$$

for $x \notin \mathbf{Z}$. Indeed, in [11], p.149, the formula

$$\pi \cot \pi z = \frac{1}{z} + \sum_{k=1}^{\infty} (\frac{1}{z+k} + \frac{1}{z-k}) \quad (4)$$

is recalled; formula (3) may be obtained by differentiating both sides of (4). Moreover, as shown in [11], p. 148, Euler's formulae for $\zeta(2n)$ follow easily from (4). To summarize, Theorem 1.1. provides an elementary probabilistic proof of (3), and therefore of Euler's formulae for $\zeta(2n)$.

Remark 5.4. We may also write formula (3) as:

$$\frac{\pi^2}{(\sin(\pi x))^2} = \sum_{k=0}^{\infty} \frac{1}{(k+x)^2} + \sum_{k=0}^{\infty} \frac{1}{(k+1-x)^2} \qquad (3)'$$

On the other hand, Binet's formula for $\Psi(z) = (\log \Gamma(z))'$ is known :

$$\Psi(z) = -\gamma + \int_0^1 \frac{1-u^{z-1}}{1-u} du$$

Starting from this classical formula, we have:

$$\Psi'(z) = -\int_0^1 \frac{u^{z-1} \log u}{1-u} du = \sum_{l=0}^{\infty} \frac{1}{(z+l)^2}.$$

Now, (3)' writes:

$$\Psi'(x) + \Psi'(1-x) = \frac{\pi^2}{\sin^2(\pi x)}$$

which is easily seen to be a consequence of the complements for the gamma function.

Remark 5.5.
We note that , under the condition: $n > m+1$,

$$\int_0^\infty \frac{x^m-1}{x^n-1}dx = \frac{\pi}{n}\frac{\sin\frac{m}{n}\pi}{\sin\frac{\pi}{n}\sin\frac{m+1}{n}\pi} = \frac{\pi}{n}(-\cot(\frac{(m+1)\pi}{n}) + \cot(\frac{\pi}{n})). \quad (5)$$

This may be obtained from $\left(\frac{\frac{\pi}{n}}{\sin\frac{m+1}{n}\pi}\right)^2 = \int_0^\infty \frac{x^m\log x}{x^n-1}dx$ by integrating both sides with respect to m since $\frac{d}{dz}(\frac{\pi}{n}\cot\frac{\pi}{n}(z+1)) = \frac{-(\frac{\pi}{n})^2}{\sin^2(\frac{\pi}{n}(z+1))}$.
Using (5), doing the same thing as before, we get:

$$\sum_{k=0}^\infty \frac{1}{(nk+1)(nk+m+1)} + \sum_{k=0}^\infty \frac{1}{(nk+n-m-1)(nk+n-1)} = \frac{\pi}{nm}\frac{\sin\frac{m}{n}\pi}{\sin\frac{\pi}{n}\sin\frac{m+1}{n}\pi} \quad (6)$$

We also note that (6) is equivalent to (4).
Remark 5.6. Putting $d_{m,n} = \frac{\sin\frac{\pi}{n}\sin\frac{(m+1)\pi}{n}}{\frac{\pi}{n}\sin\frac{m\pi}{n}}$, Consider a random variable $Y_{m,n}$with density $f_{Y_{m,n}}(x) = d_{m,n}\frac{x^m-1}{x^n-1}1_{x>0}$.
Then we have the following moments results:

$$E(Y_{m,n}^k) = d_{m,n}\int_0^\infty x^k\frac{x^m-1}{x^n-1}dx = d_{m,n}(\frac{1}{d_{k+m,n}} - \frac{1}{d_{k,n}}).$$

Remark 5.7. If we take two independent random variavles X and Y such that $f_X(x) = \frac{1}{\log(1+a)}\frac{1}{1+x}$ $(0 < x < a)$ and $f_Y(x) = \frac{1}{\log(1+b)}\frac{1}{1+x}$ $(0 < x < b)$, Then we get that $f_{XY}(x) = \frac{1}{\log(1+a)\log(1+b)(x-1)}\log\frac{(1+a)(1+b)x}{(x+a)(x+b)}$ $(0 < x < ab)$ and from this, similar computations give that the following intersting identity:

$$\begin{aligned} &-\log(1+a)\log(1+b) \\ = &\sum_{k=0}^\infty \frac{1}{(k+1)^2}((\frac{a}{1+a})^{k+1} + (\frac{b}{1+b})^{k+1} - (\frac{a(1+b)}{1+a})^{k+1} - (\frac{b(1+a)}{1+b})^{k+1} + (ab)^{k+1}). \end{aligned}$$

In the case $a = b = 1$, this identity is equivalent to

$$-(\log 2)^2 + \frac{\pi^2}{6} = 2\sum_{k=0}^\infty \frac{1}{2^{k+1}(k+1)^2},$$

which is already known (see [7]).

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